MAHLER MEASURE OF ALEXANDER POLYNOMIALS

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ABSTRACT: Let l be an oriented link of d components in a homology 3-sphere. For any nonnegative integer q, let $l(q)$ be the link of $d-1$ components obtained from l by performing $1/q$ surgery on its dth component l_d. The Mahler measure of the multivariable Alexander polynomial $\Delta_{l(q)}$ converges to the Mahler measure of Δ_l as q goes to infinity, provided that l_d has nonzero linking number with some other component. If l_d has zero linking number with each of the other components, then the Mahler measure of $\Delta_{l(q)}$ has a well defined but different limiting behaviour. Examples are given of links l such that the Mahler measure of Δ_l is small. Possible connections with hyperbolic volume are discussed.

Keywords: Mahler measure, Alexander polynomial.

1. Introduction. The *Mahler measure* of a nonzero complex Laurent polynomial f, introduced by Mahler in [**Mah60**] and [**Mah62**], is defined by

$$\mathbf{M}(f) = \exp\left(\int_{\mathbf{S}^d} \log|f(\mathbf{s})|\, d\mathbf{s}\right).$$

Here $d\mathbf{s}$ indicates integration with respect to normalized Haar measure, while $\mathbf{S}^d$ is the multiplicative d-torus, the subgroup of complex space $\mathbf{C}^d$ consisting of all vectors $\mathbf{s} = (s_1, \ldots, s_d)$ with $|s_1| = \cdots = |s_d| = 1$. We adopt the convention that the Mahler measure of the zero polynomial is 0.

It is obvious that Mahler measure is multiplicative, and the measure of any unit is 1. It is known that $M(f) = 1$ if and only if f is equal up to a unit factor to the product of cyclotomic polynomials in a single variable evaluated at monomials (see [**Sch95**, Lemma 19.1]).

The quantity $\mathbf{M}(f)$ is the geometric mean of $|f|$ over $\mathbf{S}^d$. By Jensen's formula [**Alh66**, p. 208] the Mahler measure of a nonzero polynomial $f(u) = c_n u^n + \cdots + c_1 u + c_0$ $(c_n \neq 0)$ of a single variable is

$$\mathbf{M}(f) = |c_n| \cdot \prod_{j=1}^{n} \max(|r_j|, 1),$$

where $r_1, \ldots, r_n$ are the roots of f. A short proof can be found in either [**EW99**] or [**Sch95**].

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The group ring $\mathbf{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}] \cong \mathbf{Z}\mathbf{Z}^d$ of Laurent polynomials with integer coefficients will be denoted by $\mathcal{R}_d$. It is easy to see that the set of Mahler measures of polynomials in $\mathcal{R}_1$ is contained in $\mathcal{R}_d$, for any $d > 1$. It is widely believed that the containment is proper, but no proof is known. Furthermore, if $f \in \mathcal{R}_d$, then there is no obvious relationship between the Mahler measure of f and that of the 1-variable polynomial obtained from f by setting all of its variables equal.

In [**Leh33**] Lehmer found that the Mahler measure of the degree 10 polynomial

$$L(u) = 1 + u - u^3 - u^4 - u^5 - u^6 - u^7 + u^9 + u^{10}$$

is approximately 1.17628. He subsequently asked whether, given any $\epsilon > 0$, there exists a polynomial $f \in \mathcal{R}_1$ such that $1 < M(f) < 1 + \epsilon$. Lehmer's question remains open. Despite extensive computer searches no polynomial $f \in \mathcal{R}_1$ has been found such that $1 < M(f) < M(L)$. The reader might consult Waldschmidt [**Wal80**], Boyd [**Boy81**], Stewart [**Ste78**] or Everest and Ward [**EW99**] for more about Lehmer's question.

The set $\mathcal{L} = \{M(f(u)) \mid f \in \mathcal{R}_1\}$ is a natural object for investigation. As Boyd observed in [**Boy81**], Lehmer's question is equivalent to the question of whether 1 is a limit point of $\mathcal{L}$. The *kth derived set* $\mathcal{L}^{(k)}$ is defined inductively: $\mathcal{L}^{(0)} = \mathcal{L}$, and $\mathcal{L}^{(k)}$ is the set of limit points of $\mathcal{L}^{(k-1)}$. If $1 \in \mathcal{L}$, then it can be seen from the multiplicativity of M that $\mathcal{L}^{(k)} = [1, \infty)$, for all $k \geq 1$.

The group ring $\mathcal{R}_d$ has a natural involution $^-\colon \mathcal{R}_d \to \mathcal{R}_d$, sending each u_i to u_i^{-1}. A polynomial $f \in \mathcal{R}_1$ is *reciprocal* if $\bar{f}(u) = u^n f(u)$ for some n. By a result of Smyth [**Smy71**] the Mahler measure of any nonreciprocal, irreducible polynomial $f \in \mathcal{R}_1$ not divisible by $u - 1$ is at least $1.3247\ldots$, the real root of $u^3 - u + 1$. The condition that f is not divisible by $u - 1$ is necessary, since multiplying any reciprocal polynomial by $u - 1$ yeilds a nonreciprocal polynomial with the same Mahler measure as f. As a consequence of Smyth's theorem, one need consider only reciprocal polynomials when addressing Lehmer's question.

Let $l = l_1 \cup \cdots \cup l_d$ be an oriented link of d components in the 3-sphere S^3 or more generally an orientable homology 3-sphere Σ. (A *homology 3-sphere* is a closed 3-manifold with the same integral homololgy groups as S^3.) The *exterior* E of l is the closure of Σ minus a tubular neighborhood of l. The homology group $H_1 E \cong \mathbf{Z}^d$ has natural basis represented by the meridians $m_1, \ldots, m_d$ of l with orientations induced by the link. All homology groups in this paper have integer coefficients.

The abelianization homomorphism $\gamma : \pi_1 E \to \mathbf{Z}^d$ determines a covering $p : E_\gamma \to E$, the universal abelian cover of the link exterior. The $(0th)$ *Alexander polynomial* of l, denoted here by $\Delta_l = \Delta_l(u_1, \ldots, u_d)$, is the first characteristic polynomial of the *Alexander module* $H_1(E_\gamma, p^{-1}(*))$. (In general, the *ith Alexander polynomial* is the $(i+1)$th characteristic polynomial. It is defined only up to multiplication by units in $\mathcal{R}_d$.) Alexander polynomials are easily computed from link diagrams. See [**CF63**], [**Kaw96**].

The following well-known theorem will be used repeatedly. A proof can be found in [**Kaw96**].

Theorem 1.1. (Torres conditions) The Alexander polynomial $\Delta_l(u_1, \ldots, u_d)$ of a d-component link $l = l_1 \cup \ldots \cup l_d$ satisfies:

(1) $\Delta_l(u_1, \ldots, u_d) \doteq \Delta_l(u_1^{-1}, \ldots, u_d^{-1})$;

(2) $\Delta_l(u_1, \ldots, u_{d-1}, 1) \doteq \begin{cases} \dfrac{u_1^{\lambda_1} - 1}{u_1 - 1}\, \Delta_{l'}(u_1) & \text{if } d = 2 \\ (u_1^{\lambda_1} \cdots u_{d-1}^{\lambda_{d-1}} - 1)\; \Delta_{l'}(u_1, \ldots, u_{d-1}) & \text{if } d \geq 3. \end{cases}$

Here l' denotes the link $l_1 \cup \cdots \cup l_{d-1}$ while λ_i is the linking number $\mathrm{Lk}(l_i, l_d)$. The symbol $\doteq$ indicates equality up to a unit factor.

The first Torres condition states that Alexander knot polynomials are reciprocal. In view of Smyth's result noted above, they are a natural source of examples for the study of Lehmer's question. Adding to their interest is the observation of Short and Neumann in [**Kir97**] that Lehmer's polynomial $L(u)$ is the Alexander polynomial of a knot $k \subset S^3$, and hence necessarily of infinitely many knots. One such knot is the $(-2, 3, 7)$-pretzel knot [**Hir98**].

There are other abelian covers of E besides the universal abelian one. Given a finite-index subgroup $\Lambda \subset \mathbf{Z}^d$ one can consider the finite-sheeted cover E_Λ associated to the homomorphism $\pi_1 E \xrightarrow{\gamma} \mathbf{Z}^d \to \mathbf{Z}^d/\Lambda$, where the second map is the natural projection. Such a cover can be completed to a branched cover M_Λ. In [**SW99**] we proved that for any oriented link $l \subset S^3$, the Mahler measure of Δ_l has a natural topological interpretation as the exponential rate of growth of the order of the torsion subgroup $TH_1 M_\Lambda$, computed as a suitably defined measure of Λ goes to infinity. (Although results in [**SW99**] were stated for links in S^3, they generalize easily for links in homology spheres Σ.)

In view of what has been said, it is reasonable to expect that topology—knot theory in particular—can shed some light on Lehmer's question.

2. Surgery on links and Mahler measure limits. Let k be an oriented knot in a homology 3-sphere Σ with tubular neighborhood $V = S^1 \times D^2$. The homology group $H_1 \partial V \cong \mathbf{Z}^2$ has a natural, well-defined basis represented by an oriented meridian m and a longitutde l of k. A *longitude* is a simple closed curve in ∂V that is essential but null-homologous in $\Sigma - \mathit{int}\ V$, oriented in the direction of k.

Let p, q be relatively prime integers. Removing V from Σ and reattaching it so that $* \times \partial D^2 \subset \partial V$ represents $p \cdot [\ m\ ] + q \cdot [\ l\ ] \in H_1 \partial V$ produces a 3-manfold $\Sigma' = \Sigma'(k; p/q)$ said to be *obtained from Σ by p/q surgery on k.* It is well known that Σ' is a homology 3-sphere if and only if $p = \pm 1$.

Definition 2.1. Let $l = l_1 \cup \cdots \cup l_d$ be an oriented link in a homology 3-sphere, and let q be a nonzero integer. Then $l(q)$ is the oriented link $l_1 \cup \cdots \cup l_{d-1}$ regarded in $\Sigma'(l_d; 1/q)$.

If l is a link in S^3 and l_d is unknotted, then one can describe $l(q)$ simply: Let D be a 2-disk that bounds l_d. We can assume that the sublink $l_1 \cup \cdots \cup l_{d-1}$ intersects D transversely. Then $l(q)$ is the link in S^3 obtained from $l_1 \cup \cdots \cup l_{d-1}$ by cutting the strands that pass through D, twisting q full times in the direction of the longitude of l_d, and then reconnecting. (Details can be found in [**Rol76**].) Kidwell investigated this case in [**Kid82**]. He showed that if l_d has nonzero linking number with some other component of l, then, as q goes to infinity, the degree of the "reduced Alexander polynomial" of $l(q)$, the polynomial obtained from $\Delta_{l(q)}$ by setting $u_1 = \cdots = u_{d-1}$, grows without bound. He found, in fact, that the sequence of nonzero exponents of the reduced Alexander polynomial acquires an ever-expanding gap. In contrast, Theorem 2.2 asserts that the Mahler measures of the Alexander polynomials are well behaved.

Theorem 2.2. Assume that $l = l_1 \cup \cdots \cup l_d$ is an oriented link in a homology 3-sphere, and let $l(q)$ be as in Definition 2.1. If some linking number $\lambda_i = \mathrm{Lk}(l_i, l_d)$ is nonzero, $1 \le i \le d-1$, then

$$\lim_{q\to\infty} M(\Delta_{l(q)}) = M(\Delta_l).$$

Remarks 2.3. 1. By replacing $1/q$ with p/q in Theorem 2.2 more general, albeit more complicated, results are possible. For the sake of simplicity we have chosen not to work in such generality.

2. Theorem 2.2 bears a striking resemblance of form to a theorem of Thurston [**Thu83**] which states that the volume of a hyperbolic 3-manifold with cusps is the limit of the volumes of the manifolds obtained by performing (p_i, q_i) Dehn surgery on the ith cusp. The pairs (p_i, q_i) are required to go to infinity in a suitable manner. (See also [**NZ85**].) We are grateful to John Dean for bringing this to our attention.

In [**Kid82**] Kidwell did not address the case in which l_d has zero linking number with each of the remaining components of l. Theorem 2.4 completes the picture.

Theorem 2.4. Assume that $l = l_1 \cup \cdots \cup l_d$ is an oriented link in a homology 3-sphere.

(1) If $\lambda_i = 0$, for each $1 \le i \le d-1$, then

$$\lim_{q\to\infty} \frac{1}{q}\, \Delta_{l(q)}(u_1, \cdots, u_{d-1}) \doteq \begin{cases} (u_1 - 1)\, \dfrac{\partial}{\partial u_2}\Big|_{u_2=1} \Delta_l(u_1, u_2), & \text{if } d = 2; \\ \dfrac{\partial}{\partial u_d}\Big|_{u_d=1} \Delta_l(u_1, \ldots, u_d), & \text{if } d \ge 3. \end{cases}$$

The convergence is in the strong sense that the polynomials $\Delta_{l(q)}$ have eventually constant degree, and the coefficients of $\frac{1}{q}\Delta_{l(q)}$ converge to those of the polynomial on the right.

(2) If $(u_d-1)^2$ divides $\Delta_l(u_1,\ldots,u_d)$, then for every q,

$$\Delta_{l(q)}(u_1,\ldots,u_{d-1}) \;=\; \Delta_{l'}(u_1,\ldots,u_{d-1}),$$

where $l' = l_1 \cup \cdots \cup l_{d-1}$.

Remark 2.6. The hypothesis of Theorem 2.4 that $\lambda_i = 0$ for each i implies that $\Delta_l(u_1,\ldots,u_{d-1},1) = 0$ and thus $u_d - 1$ divides Δ_l, by the second Torres condition. Consequently, partial differentiation of Δ_l with respect to u_d followed by evaluation at $u_d = 1$ is equivalent to dividing Δ_l by $u_d - 1$ and then setting u_d equal to 1 in the result. In particular, the operation is well defined even though Δ_l is determined only up to multiplication by a unit in $\mathcal{R}_d$.

By a theorem of Boyd [**Boy98**] the Mahler measure $M(f)$ is a continuous function of the coefficients of f for polynomials of fixed total degree. This yields the following corollary to Theorem 2.4.

Corollary 2.5. Under the hypothesis of Theorem 2.4,

$$\lim_{q\to\infty} \frac{1}{q}\, M(\Delta_{l(q)}) \;=\; M\left[\left.\frac{\partial}{\partial u_d}\right|_{u_d=1} \Delta_l(u_1,\ldots,u_d)\right].$$

Moreover, if $(u_d-1)^2$ divides Δ_l, then for every q,

$$M(\Delta_{l(q)}) \;=\; M(\Delta_{l'}).$$

3. Proof of Theorem 2.2. By a result proved for a special case by Boyd [**Boy81′**] and in general by Lawton [**Law83**], the Mahler measure of any polynomial $f \in \mathcal{R}_d$ can be expressed as the limit of Mahler measures of polynomials in a single variable. For $\mathbf{r} = (r_1,\ldots,r_d)$ and $\mathbf{n} = (n_1,\ldots,n_d) \in \mathbf{Z}^d$ define

$$|\mathbf{n}| = \max\{|n_1|,\ldots,|n_d|\},$$

$$(\mathbf{r},\mathbf{n}) = r_1n_1 + \cdots + r_dn_d,$$

and

$$\langle \mathbf{r}\rangle = \min\{|\mathbf{n}| : \mathbf{0} \neq \mathbf{n} \in \mathbf{Z}^d, (\mathbf{r},\mathbf{n}) = 0\}.$$

Also, let

$$f_{\mathbf{r}}(u) = f(u^{r_1}, \ldots, u^{r_d}).$$

Lemma 3.1. [Boyd-Lawton Lemma] For every $f \in \mathcal{R}_d$, $\quad \lim_{\langle \mathbf{r} \rangle \to \infty} M(f_{\mathbf{r}}) = M(f)$.

We also need the following consequence of Lemma 3.1.

Corollary 3.2. Let $\kappa_1, \ldots, \kappa_{d-1}$ be integers, not all zero. For any $f \in \mathcal{R}_d$ and positive integer q, let $f^{(q)}$ be the element of $\mathbf{R}_{d-1}$ defined by

$$f^{(q)}(u_1, \ldots, u_{d-1}) = f(u_1, \ldots, u_{d-1},\ (u_1^{\kappa_1} \cdots u_{d-1}^{\kappa_{d-1}})^q).$$

Then $\lim_{q \to \infty} M(f^{(q)}) = M(f)$.

Proof. By Lemma 3.1, given $\epsilon > 0$, there exists $K > 0$ such that $|M(f_{\mathbf{s}}) - M(f)| < \epsilon/2$ whenever $\mathbf{s} \in \mathbf{Z}^d$ satisfies $\langle \mathbf{s} \rangle \geq K$. We will show that for $q \geq K$ we have $|M(f^{(q)}) - M(f)| < \epsilon$, from which Corollary 3.2 follows.

Fix $q \geq K$. Choose $\mathbf{r} \in \mathbf{Z}^{d-1}$ such that $\langle \mathbf{r} \rangle \geq 2Kq \cdot \max_i \{\kappa_i\}$, and also such that

$$|M(f_{\mathbf{r}}^{(q)}) - M(f^{(q)})| < \epsilon/2.$$

Set $\mathbf{r}^+ = (r_1, \ldots, r_{d-1}, q(\kappa_1 r_1 + \cdots + \kappa_{d-1} r_{d-1}))$. Then $f_{\mathbf{r}}^{(q)} = f_{\mathbf{r}^+}$, so it suffices to show that $\langle \mathbf{r}^+ \rangle \geq K$. Suppose $(\mathbf{n}, \mathbf{r}^+) = 0$, where $\mathbf{0} \neq \mathbf{n} = (n_1, \ldots, n_d)$. Then

$$\sum_{i=1}^{d-1} (n_i + n_d q \kappa_i) r_i = 0.$$

Hence either (i) $n_i + n_d q \kappa_i = 0$, for all i; or else (ii) $|n_j + n_d q \kappa_j| \geq \langle \mathbf{r} \rangle \geq 2Kq \cdot \max_i |\kappa_i|$, for some j. In case (i) we must have $n_d \neq 0$. Since we assume that $\kappa_k \neq 0$ for some k, it follows that $|n_k| \geq q \geq K$ and so $|\mathbf{n}| \geq K$. In case (ii) either $|n_j|$ or $|n_d| \cdot q \cdot |\kappa_j|$ is at least $K \cdot q \cdot \max_i |\kappa_i|$, and so again we have $|\mathbf{n}| \geq K$. ∎

Proof of Theorem 2.2. Our proof closely follows a proof of the second Torres condition found in [**Kaw96**].

Let E' denote the exterior of $l(q)$ in the homology 3-sphere $\Sigma' = \Sigma'(l; l_d, 1/q)$. Let

$$\gamma : \pi_1 E' \to H_1 E' \cong \langle u_1, \ldots, u_{d-1}\ :\ [u_i, u_j] = 1\ (1 \leq i < j < d) \rangle$$

be the abelianization homomorphism mapping the class of the ith meridian m_i to u_i, $1 \leq i < d$, and mapping the class of m_d to $u_1^{-q\lambda_1} \cdots u_{d-1}^{-q\lambda_{d-1}}$. We will denote the corresponding covering space by $p : E'_\gamma \to E'$.

The exterior E of the original link l is a subspace of E'. Let ν be the natural composite epimorphism $\pi_1 E \to \pi_1 E' \xrightarrow{\gamma} \mathbf{Z}^{d-1}$. The total space of the corresponding cover $E_\nu \to E$ can be identified with the preimage $p^{-1}(E) \subset E'_\gamma$.

Consider the portion of the homology exact sequence of the pair (E'_γ, E_ν):

$$\cdots \to H_2 E'_\gamma \xrightarrow{j_*} H_2(E'_\gamma, E_\nu) \xrightarrow{\partial_*} H_1 E_\nu \xrightarrow{i_*} H_1 E'_\gamma \xrightarrow{j_*} H_1(E'_\gamma, E_\nu) \to \cdots \tag{3.1}$$

Here all homology groups may be regarded as $\mathcal{R}_{d-1}$-modules. By the excision isomorphism $H_k(E'_\gamma, E_\nu)$ is trivial unless $k = 2$, and $H_2(E'_\gamma, E_\nu) \cong \mathcal{R}_{d-1}/(u_1^{\lambda_1} \cdots u_{d-1}^{\lambda_{d-1}} - 1)$. Hence we have a short exact sequence

$$0 \to ker\ i_* \to H_1 E_\nu \to H_1 E'_\gamma \to 0.$$

The $0th$ characteristic polynomials satisfy

$$\Delta_0(H_1 E_\nu)\ \doteq\ \Delta_0(ker\ i_*)\ \Delta_0(H_1 E'_\gamma). \tag{3.2}$$

As in [**Kaw96**, Proposition 7.3.10], we find that $\Delta_0(H_1 E'_\gamma)$ is the Alexander polynomial of $l(q)$, while

$$\Delta_0(H_1 E_\nu)\ \doteq\ \begin{cases} \Delta_l^{(q)} & \text{if } d \geq 3 \\ (u_1 - 1)\ \Delta_l^{(q)} & \text{if } d = 2. \end{cases}$$

(Here we use the notation of Corollary 3.2 with $\kappa_i = -\lambda_i,\ 1 \leq i < d$.) Since $\Delta_0(ker\ i_*)$ is a divisor of $\Delta_0(H_2(E'_\gamma, E_\nu)) \doteq u_1^{\lambda_1} \cdots u_{d-1}^{\lambda_{d-1}} - 1$, its Mahler measure is 1. Hence the Mahler measure of $\Delta_{l(q)}$ is equal to that of $\Delta_l^{(q)}$. Theorem 2.2 now follows from Corollary 3.2. ∎

Remark 3.3. The equation (3.2) can be improved. We have

$$\Delta_l^{(q)}(u_1, \ldots, u_{d-1})\ \doteq\ \begin{cases} (u_1^{\lambda_1} \cdots u_{d-1}^{\lambda_{d-1}} - 1)\ \Delta_{l(q)} & \text{if } d \geq 3 \\ \dfrac{u_1^{\lambda_1} - 1}{u_1 - 1}\ \Delta_{l(q)} & \text{if } d = 2. \end{cases} \tag{3.3}$$

If $H_2(E'_\gamma) = 0$, then (3.3) follows immediately from (3.1) and (3.2). If $H_2(E'_\gamma) \neq 0$, then as in [**Kaw96**, 7.3.5] we have $\Delta_0(H_1 E'_\gamma) = 0$; in this case both sides of (3.2) vanish, and (3.3) is trivial.

Equation (3.3) can also be obtained by applying Theorem 6.7 of [**Fox60**].

4. Proof of Theorem 2.4. Following [**Kid82**] we add an unknotted oriented component l_{d+1} to $l = l_1 \cup \cdots \cup l_d$ such that $Lk(l_d, l_{d+1}) = Lk(l_{d-1}, l_{d+1}) = 1$, while $\mathrm{Lk}(d_i, d_{i+1}) = 0$ for $1 \leq i < d-1$. We denote the augmented link by l^+. By the second Torres condition,

$\Delta_{l^+}(u_1,\ldots,u_d,1) \doteq (u_{d-1}u_d-1)\,\Delta_l(u_1,\ldots,u_d)$. Differentiating each side of the equation and recalling that $\Delta_l(u_1,\ldots,u_d)=0$, we have

$$\frac{\partial}{\partial u_d}\bigg|_{u_d=1} \Delta_{l^+}(u_1,\ldots,u_d,1) \;\doteq\; (u_{d-1}-1)\,\frac{\partial}{\partial u_d}\bigg|_{u_d=1} \Delta_l(u_1,\ldots,u_d). \qquad (4.1)$$

Equation (3.3) implies that

$$\Delta_{l^+}(u_1,\ldots,u_{d-1},u_{d+1}^q,u_{d+1}) \;\doteq\; (u_{d+1}-1)\,\Delta_{l^+(q)}(u_1,\ldots,u_{d-1},u_{d+1}),$$

where $l^+(q)$ is the d-component link obtained from l^+ by performing $1/q$-surgery on the component l_d. Again by differentiating, and applying the second Torres condition, we have

$$\begin{aligned}\frac{\partial}{\partial u_{d+1}}\bigg|_{u_{d+1}=1} \Delta_{l^+}(u_1,\ldots,u_{d-1},u_{d+1}^q,u_{d+1}) \;&\doteq\; \Delta_{l^+(q)}(u_1,\ldots,u_{d-1},1)\\ &\doteq\; \begin{cases}\Delta_{l(q)}(u_1) & \text{if } d=2\\ (u_{d-1}-1)\,\Delta_{l(q)}(u_1,\ldots,u_{d-1}) & \text{if } d\ge 3.\end{cases}\end{aligned} \qquad (4.2)$$

Comparing (4.1) and (4.2) we see that in order to prove the first assertion of Theorem 2.4 it suffices to show that

$$\lim_{q\to\infty}\frac{1}{q}\,\frac{\partial}{\partial u_{d+1}}\bigg|_{u_{d+1}=1} \Delta_{l^+}(u_1,\ldots,u_{d-1},u_{d+1}^q,u_{d+1}) \;\doteq\; \frac{\partial}{\partial u_d}\bigg|_{u_d=1} \Delta_{l^+}(u_1,\ldots,u_d,1). \quad (4.3)$$

By collecting terms, we can write the Alexander polynomial of Δ_{l^+} in the form

$$\Delta_{l^+}(u_1,\ldots,u_{d+1}) \;\doteq\; \sum_{i=0}^{m}\sum_{j=0}^{n} f_{ij}(u_1,\ldots,u_{d-1})\;u_d^i\;u_{d+1}^j$$

for suitable $f_{ij}\in\mathcal{R}_{d-1}$. A simple calculation shows that

$$\frac{\partial}{\partial u_{d+1}}\bigg|_{u_{d+1}=1} \Delta_{l^+}(u_1,\ldots,u_{d-1},u_{d+1}^q,u_{d+1}) \;\doteq\; \sum_{i=0}^{m}\sum_{j=0}^{n} f_{ij}\cdot(qi+j)$$

and

$$\frac{\partial}{\partial u_d}\bigg|_{u_d=1} \Delta_{l^+}(u_1,\ldots,u_d,1) \;\doteq\; \sum_{i=0}^{m}\sum_{j=0}^{n} f_{ij}\cdot i,$$

so that (4.3) is immediate.

Now suppose that $(u_d-1)^2$ divides Δ_l. Then from (4.1),

$$\sum_{i=0}^{m}\sum_{j=0}^{n} f_{ij}\cdot i \;\doteq\; \frac{\partial}{\partial u_d}\bigg|_{u_d=1} \Delta_{l^+}(u_1,\ldots,u_d,1)=0.$$

Thus equation (4.3) bcomes

$$\sum_{i=0}^{m}\sum_{j=0}^{n} f_{ij}\cdot j \ \doteq\ \begin{cases} \Delta_{l(q)}(u_1) & \text{if } d=2 \\ (u_{d-1}-1)\ \Delta_{l(q)}(u_1,\ldots,u_{d-1}) & \text{if } d\geq 3. \end{cases}$$

Let $\tilde{l} \ = \ l_1\cup\cdots\cup l_{d-1}\cup l_{d+1}$. By the second Torres condition:

$$\Delta_{l^+}(u_1,\ldots,u_{d-1},1,u_{d+1}) \ \doteq\ (u_{d+1}-1)\ \Delta_{\tilde{l}}(u_1,\ldots,u_{d-1},u_{d+1}).$$

Differentiating, we have

$$\Delta_{\tilde{l}}(u_1,\ldots,u_{d-1},1) \ \doteq\ \frac{\partial}{\partial u_{d+1}}\bigg|_{u_{d+1}=1} \Delta_{l^+}(u_1,\ldots,u_{d-1},1,u_{d+1}) \ \doteq\ \sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}\cdot j.$$

Hence

$$\Delta_{\tilde{l}}(u_1,\ldots,u_{d-1},1) \ \doteq\ \begin{cases} \Delta_{l(q)}(u_1) & \text{if } d=2 \\ (u_{d-1}-1)\Delta_{l(q)}(u_1,\ldots,u_{d-1}) & \text{if } d\geq 3. \end{cases} \tag{4.4}$$

On the other hand,

$$\Delta_{\tilde{l}}(u_1,\ldots,u_{d-1},1) \ \doteq\ \begin{cases} \Delta_{l'}(u_1) & \text{if } d=2 \\ (u_{d-1}-1)\Delta_{l'}(u_1,\ldots,u_{d-1}) & \text{if } d\geq 3, \end{cases} \tag{4.5}$$

again using the second Torres condition. Comparing (4.4) and (4.5) we are done.∎

Examples illustrating Theorem 2.2 can be found in section 5. We conclude this section with two examples that illustrate Theorem 2.4 and its corollary.

The following lemma of Boyd [**Boy81**] is often useful when computing Mahler measures of Alexander polynomials. A proof can also be found in [**Sch95**, p.157].

Lemma 4.1. If $g\in\mathcal{R}_{d+1}$ is defined by

$$g(u_1,\ldots,u_d,u_{d+1}) = u_{d+1}f(u_1,\ldots,u_d) + f(u_1^{-1},\ldots,u_d^{-1}),$$

for $f\in\mathcal{R}_d$, then $M(g)=M(f)$.

Example 4.2. Consider the Whitehead link $l=5_1^2$ in Figure 1. Its Alexander polynomial is $(u_1-1)(u_2-1)$. The knots $l(q)$ are often called "twist knots," and their Alexander polynomials $qu_1^2-(2q+1)u_1+q$ are easily computed. The Mahler measure of $\Delta_{l(q)}$ increases without bound as q goes to infinity. However,

$$\frac{1}{q}M(\Delta_{l(q)}) = M\left(u_1^2 - \frac{2q+1}{q}\ u_1+1\right)$$

approaches $M(u_1^2 - u_1 + 1) = 1 = M(\Delta_l)$, as predicted by Corollary 2.5.

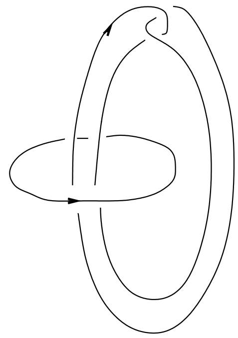


Figure 1: Whitehead link $l = 5_1^2$

Example 4.3. Consider the 3-component link 9_8^3 in Figure 2. Its Alexander polynomial is $(u_2 - 1)(u_3 - 1)(u_1 + 2u_2 - 2u_1u_2 - u_2^2)$. Differentiating, we find

$$\frac{\partial}{\partial u_3}\bigg|_{u_3=1} \Delta_l(u_1, u_2, u_3) \doteq (u_2 - 1)(u_1 + 2u_2 - 2u_1u_2 - u_2^2),$$

which can be rewritten as $(u_2 - 1)[2u_2 - u_2^2 - u_1u_2^2(2u_2^{-1} - u_2^{-2})]$. Using Lemma 4.1 and a change of variable we see that

$$M\left[\frac{\partial}{\partial u_3}\bigg|_{u_3=1} \Delta_l(u_1, u_2, u_3)\right] = M[2u_2 - u_2^2] = 2.$$

By Corollary 2.5 the Mahler measure of $\Delta_{l(q)}$ is asymptotic to $2q$.

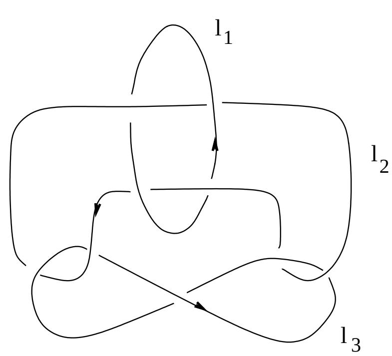


Figure 2: The link $l = 9_8^3$

Examples of d-component links l that have Alexander polynomials divisible by $(u_d - 1)^2$ are not difficult to find. Perhaps the simplest example is $l = 8_{10}^2$. (Here the component l_1 is unknotted.) The reader interested in an exercise can draw the knots $l(q)$ and verify that their Alexander polynomials are trivial.

5. Alexander link polynomials with small Mahler measure.

Example 5.1. Consider the 2-component link $l = 7_1^2$ in Figure 3a. Its Alexander polynomial $1 - u_1 + (-1 + u_1 - u_1^2)u_2 + (-u_1 + u_1^2)u_2^2$ can be represented schematically:

$$\begin{array}{ccc} & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & \end{array}$$

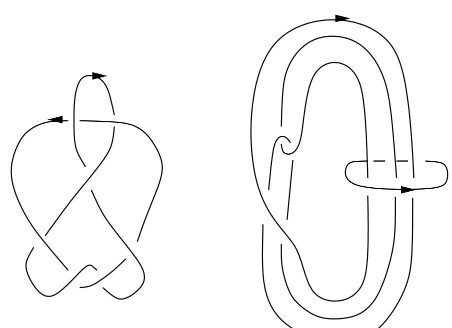
Here the number in the $(i+1)$th column from the left and the $(j+1)$th row from the bottom is the coefficient of $u_1^i u_2^j$.

Replacing u_i by $-u_i$, for $i = 1, 2$, a change that leaves the Mahler measure unaffected, produces $1 + u_1 + (1 + u_1 + u_1^2)u_2 + (u_1 + u_1^2)u_2$:

$$\begin{array}{ccc} & +1 & +1 \\ +1 & +1 & +1 \\ +1 & +1 & \end{array}.$$

Boyd [**Boy78**] has computed the Mahler measure as approximately 1.25543. It is the smallest known value the derived set of $\mathcal{L}^{(1)}$ (see [**Boy78**].)

The link l is redrawn in Figure 3b so that the second component l_2 appears as a standard unknotted circle. The knots $l(q) \subset S^3$ are now easy to visualize: they are obtained from l_1 by giving q full right-hand twists to the strands passing through l_2. By the proof of Theorem 2.2, $\Delta_{l(q)}$ has the same Mahler measure as $\Delta_l(u, u^q)$. When $q = 11$, we obtain Lehmer's value $M(L(u))$. When $q = 10$ we get the value $1.18836\ldots$, the second smallest known value of $\mathcal{L}$ that is greater than 1.



a) b)

Figure 3: The link 7_1^2

Example 5.2. The link 6_2^2 is shown in Figure 2. Its Alexander polynomial $u_1 + (1 - u_1 + u_1^2)u_2 + u_1 u_2^2$:

$$\begin{array}{ccc} & +1 & \\ +1 & -1 & +1 \\ & +1 & \end{array}$$

has Mahler measure $1.28573\ldots$, which is the second smallest known value of $\mathcal{L}^{(1)}$ (see [**Boy78**]). The links $9^2_4, 9^2_9$ and 9^2_{50} also have Alexander polynomials with this Mahler measure.

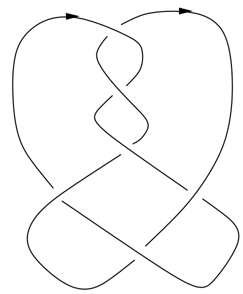


Figure 4: The link 6^2_2

Example 5.3. Using a computer search, Mossinghoff [**Mos98**] discovered the polynomial $u_1^2u_2(u_2+1) + u_1(u_2^4 - u_2^2 + 1) + u_2^2(u_2+1)$:

$$\begin{array}{ccc} & +1 & \\ +1 & & \\ +1 & -1 & +1 \\ & & +1 \\ & +1 & \end{array}$$

which has Mahler measure approximately 1.30909. This value is the third smallest known value of $\mathcal{L}^{(1)}$. In a recent private communication Mossinghoff observed that the symmetric polynomial

$$\Delta(u_1, u_2) \doteq u_1^{-2}u_2^{-2} + u_1^{-1} - u_2^{-1} - 1 + u_1 - u_2 + u_1^2u_2^2,$$

schematically:

$$\begin{array}{ccccc} & & & & +1 \\ & & -1 & & \\ & +1 & -1 & +1 & \\ & & -1 & & \\ +1 & & & & \end{array}$$

has the same small Mahler measure. By a theorem of Levine [**Lev67**] Δ is the Alexander polynomial of a (nonunique) 2-component link l in the 3-sphere.

Example 5.4. Figure 5 displays the 3-component link $l = 6^3_1$. Its Alexander polynomial Δ_l is $u_1 + u_2 + u_3 - u_1u_2 - u_1u_3 - u_2u_3$. We can rewrite the polynomial as $u_1 + u_2 - u_1u_2 - u_1u_2u_3(u_1^{-1} + u_2^{-1} - u_1^{-1}u_2^{-1})$. We replace $-u_1u_2u_3$ by u_3, a change of variables that does not affect Mahler measure. Lemma 4.1 implies that $M(\Delta_l)$ is equal to the Mahler measure of $u_1^{-1} + u_2^{-1} - u_1^{-1}u_2^{-1}$. Now replacing each u_i by $-u_i$ and multiplying by the unit $-u_1u_2$

produces the relatively simple polynomial $1 + u_1 + u_2$ with the same Mahler measure as Δ_l.

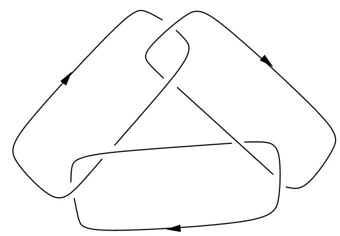


Figure 5: The link $l = 6_1^3$.

Smyth (see [**Boy81**]) has shown that the Mahler measure of $1 + u_1 + u_2$ is

$$\exp \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2},$$

where $\chi(n)$ is the Legendre symbol

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 3; \\ -1 & \text{if } n \equiv 2 \bmod 3; \\ 0 & \text{if } n \equiv 0 \bmod 3\ . \end{cases}$$

(See also [**EW99**, p. 55] for the calculation.) The Mahler measures of many polynomials turn out to be simple multiples of L-series of this sort. Deninger [**Den97**] offers a conjectural explanation in terms of K-theory.

The value of $M(\Delta_l)$, approximately 1.38135, is the smallest known element of $\mathcal{L}^{(2)}$. Applying Theorem 2.2. twice one can approximate it by Mahler measures of Alexander polynomials of knots in S^3. Beginning with l, we twist q_1 times about l_3, producing the 2-component link $l(q_1) = l_1' \cup l_2'$. Notice that the components of l_1', l_2' are each unknotted and they have nonzero linking number (equal to $+1$). Now we twist q_2 times about l_2', obtaining a knot $k = k(q_1, q_2)$. By Theorem 2.2 the limit

$$\lim_{q_1 \to \infty} \lim_{q_2 \to \infty} M(\Delta_k)$$

is equal to the Mahler measure $M(\Delta_l)$ of the link $l = 6_1^3$. The knot k is the closure ot the $(q_1 + 1)$-braid $\sigma_1^{-1}\sigma_2 \cdots \sigma_{q_1} c^{q_2}$, where $\sigma_1, \ldots, \sigma_{q_1}$ are the usual braid generators, and c is a full right-hand twist $(\sigma_1 \cdots \sigma_{q_1})^{q_1+1}$.

Example 5.5. Consider the 4-component link $l = 8_2^4$ in Figure 6. Its Alexander polynomial can be put into the form $1 - u_1 - u_2 + u_2u_3 + u_1u_2u_3u_4^{-1}(1 - u_1^{-1} - u_2^{-1} + u_2^{-1}u_3^{-1})$. By Lemma 4.4 the Mahler measure of Δ_l is equal to that of $1 - u_1 - u_2 + u_2u_3$. We replace $-u_1, -u_2, u_2u_3$ by u_1, u_2, u_3, respectively, a change of variables that does not affect Mahler

measure. We then find that the Mahler measure of Δ_l is equal to that of $1+u_1+u_2+u_3$. By a result of Smyth (reported in [**Boy81**]) this value is precisely

$$\exp \frac{7}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Until recently, this was the only nontrivial Mahler measure of a 3-variable polynomial that was evaluated in closed form (see [**Smy00**]). Its numerical value is approximately 1.53154.

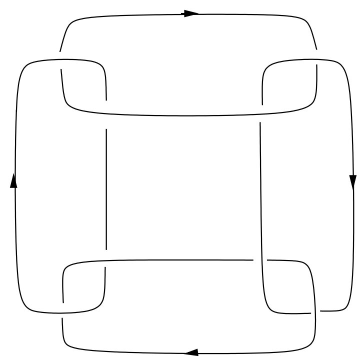


Figure 6: The link $l = 8^4_2$.

The links 6^3_1 and 8^4_2 are examples of pretzel links. For any integers $p_1 \ldots, p_d$, the *pretzel link* $l(p_1, \ldots, p_d)$ is the boundary of the surface consisting of two disks joined by d twisted vertical bands, as in Figure 5. The ith band contains $|p_i|$ half-twists, right-handed if p_i is positive, left-handed otherwise. The links 6^3_1 and 8^4_2 are $l(2,2,2)$ and $l(2,2,2,-2)$, respectively.

The Alexander polynomial Δ_l of the pretzel link $l = l(2,-2,2,-2,2)$ (see Figure 5) has the form $f + v\bar{f}$, where $f(u_1,u_2,u_3,u_4) = u_1 - u_1u_3 - u_1u_4 - u_2u_3 + u_3u_4 + u_1u_2u_3$ and $v = -u_1u_2u_3u_4u_5$. By Lemma 4.4 the Mahler measure of Δ_l is equal to that of f. Dividing f by u_1 produces $1-u_3-u_4+u_2u_3+u_1^{-1}u_3u_4-u_1^{-1}u_2u_3$. A further substitution, replacing $-u_3, -u_4, u_2u_3, u_1^{-1}u_3u_4$ by v_1, v_2, v_3, v_4, respectively, yields $1 + v_1 + v_2 + v_3 + v_4 - v_1^{-1}v_2^{-1}v_3v_4$, which has the same Mahler measure as f and hence as Δ_l.

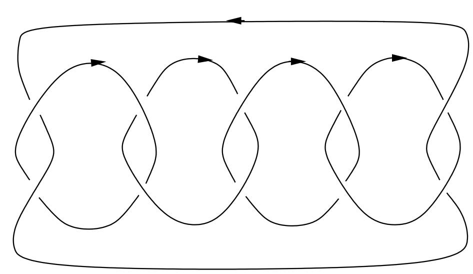


Figure 7: The pretzel link $l(2,-2,2,-2,2)$

The values of $M(1+u_1+u_2+u_3+u_4)$ and $M(\Delta_l) = M(1+v_1+v_2+v_3+v_4-v_1^{-1}v_2^{-1}v_3v_4)$ are very close: our calculations suggest that the first is $1.723\ldots$, while the second is approximately 1.729.

The link $l = l(2,-2,2,-2,2)$ has arisen recently in investigations of C. C. Adams [**Ada00**] as well as the work of N. M. Dunfield and W. P. Thurston [**Dun00**]. Adams has shown that $S^3 - l$ has the largest cusp density possible for a cusped hyperbolic 3-manifold. P. J. Callahan, C. D. Hodgson and J. R. Weeks have observed that $S^3 - l$ is obtained from most of the small-volume hyperbolic 3-manifolds of the census [**HW89**] by removing a shortest-length geodesic and then repeating the process four additional times.

Smyth and Myerson [**SM82**] have shown that $M(1+u_1+u_2+\cdots+u_n)$ is asymptotic to $e^{-C_0\sqrt{n}}$, where C_0 is Euler's constant. Boyd [**Boy00**] has asked whether this value is the rate of growth of $\min \mathcal{L}^{(n)}$.

Question 5.6. Does there exist a sequence l_n of n-component links such that the sequence of Mahler measures$M(\Delta_{l_n})$ is asymptotic to $\min \mathcal{L}^{(n)}$?

Definition 5.7. A number $\theta > 1$ is a *Pisot-Vijayaraghavan number*, or simply a *PV number*, if it is the root of a monic irreducible polynomial with integer coefficients such that all of its other roots lie strictly inside the unit disk. If some root lies on the circle but no other root is outside, then θ is a *Salem number*.

In [**Sal44**] R. Salem proved that the set of PV numbers is closed, and hence nowhere dense as it is countable. Since 1 is not a PV number, it follows at once that there is a minimum PV number θ_0. The value of θ_0 was shown by C. L. Siegel in [**Sie44**] to be the real root of $u^3 - u - 1$, approximately 1.32471.

Hironaka proved that among the reduced Alexander polynomials of pretzel knots and links $l(p_1, p_2, \cdots, p_k, -1, \cdots, -1)$, where $p_1, \ldots, p_k$ are positive and -1 occurs $k-2$ times, Mahler measure is minimized by the Lehmer polynomial $L \doteq \Delta_{l(-2,3,7)}$. (Recall that the reduced Alexander polynomial of an oriented link l is the polynomial of a single variable obtained from Δ_l by setting all of the variables equal.) A proof can be found in [**Hir98**] (see also [**GH99**]).

Example 5.8. The torus knot 5_1 is equivalent to $l(-2,3,1)$. Consider the 2-component link l obtained from it by encircling the third band of the pretzel knot, as in Figure 8. Its Alexander polynomial Δ_l is $u_1^2 - u_1^3 + u_1^5 + u_2(1 - u_1^2 + u_1^3)$. Using Lemma 4.1, one easily sees that Δ_l has the same Mahler measure as $u^3 - u - 1$, namely $\theta_0(\approx 1.32471)$.

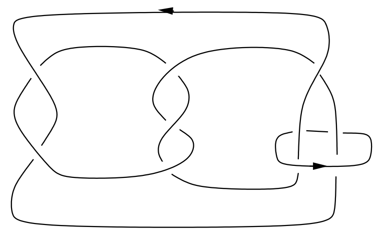


Figure 8: Encircled pretzel link

We twist the two encircled arcs of k, forming knots $l(q)$ as in section 2. The Mahler measures $M(\Delta_{l(q)})$, which converge to θ_0 by Theorem 2.2, must all be Salem numbers in view of [**Hir98**, Proposition 3.1]. Some of the values are given below.

Number of Twists	Salem Number
1	1
2	1
3	1.17628...
4	1.26123...
5	1.29348...
6	1.30840...
7	1.31591...
8	1.31986...
9	1.32201...
10	1.32319...
11	1.32385...
12	1.32423...

As we remarked in section 2, Theorem 2.2 bears a resemblance of form to a theorem of Thurston about volumes of hyperbolic manifolds. In their census of the simplest hyperbolic knots [**CDW98**], P. Callahan, J. Dean and J. Weeks record the fact that the pretzel knot $l(-2, 3, 7)$ has a complement composed of only 3 ideal tetrahedra, thereby qualifying it for the honor of second simplest hyperbolic knot, after the figure eight knot 4_1 and alongside the knot 5_2.

Question 5.9. Is there a significant relationship between the Mahler measure of the Alexander polynomial of a hyperbolic link and the hyperbolic volume of its exterior?

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